Eur. Phys. J. B (2017) 90: 160
DOI: 10.1140/epjb/e2017-80241-2

THE EUROPEAN
PHYSICAL JOURNAL B

Colloquium

Finite size and boundary effects in critical two-dimensional free-fermion models

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Received 25 April 2017 / Received in final form 23 June 2017
Published online 28 August 2017 –

Abstract. Here we will consider the finite-size scaling, finite-size corrections and boundary effects for the critical two-dimensional free-fermion models. A short review of significant achievements and possibilities is given. However, this review is still far from completeness. We derive the exact finite-size corrections for the set of free models of statistical mechanics, including Ising model, dimer model, resistor network and spanning tree model under different boundary conditions. We have shown that the partition functions of all these models can be written in terms of the only object, namely, the partition function with twisted boundary conditions.

1 Introduction

It is well known that the singularities in thermodynamic functions associated with a critical point occur only in the thermodynamic limit when dimension L of the system under consideration tends to infinity. In such a limit, the critical fluctuations are correlated over a distance of the order of correlation length ξ_{bulk} that may be defined as the length scale governing the exponential decay of correlation functions. Besides these two fundamental lengths, L and ξ_{bulk}, there is also the microscopic length of interactions a. Thermodynamic quantities thus may in principle depend on the dimensionless ratios ξ_{bulk}/L and a/L. The finite-size scaling (FSS) hypothesis [1] assumes that in the scaling interval, for temperatures so close to the critical point that $a \ll \xi_{\text{bulk}} \sim L$, the microscopic length drops out and the behavior of any thermodynamic quantity can be described in terms of the universal scaling function of the scaling variable $t = L/\xi_{\text{bulk}}$. However, non-universal corrections to FSS do exist. These sometimes can be viewed as asymptotic series in powers of a/L.

In the study of phase transitions and critical phenomena, it is extremely important to understand finite-size corrections to thermodynamical quantities. Finite-size scaling [1–3] concerns the critical behavior of systems in which one or more directions are finite, even though microscopically large and is valuable in the analysis of experimental and numerical data in many situations, for example, for films of finite thickness. Finite-size scaling theory, initiated more than four decades ago by Fisher and Barber [4] has advanced considerably during the past decades [2,5,6]. It has been found that critical systems can be classified into different universality classes so that the systems in the same class have the same set of critical exponents, whose values depends only on the global properties of the system such as spacial dimensions, number of components of the order parameter, the range of interaction and the symmetry group. The hypothesis of universality has much stronger implications and it is possible to show that models belonging to the same universality class also share the same set of universal finite-size scaling functions and amplitude ratios, whose values are independent of the microscopic structure of interactions. As soon as one has a finite system one must consider the question of boundary conditions on the outer surfaces or "walls" of the system. As is well known, the critical behavior near boundaries normally differs from the bulk behavior. In general, each bulk universality class of critical phenomena splits into several surface universality classes. The systems under various boundary conditions have the same per-site free energy in the bulk limit, whereas the finite size corrections are different. To understand the effects of boundary conditions on finite-size corrections, it is valuable to study model systems, especially those which have exact results, where the analysis can be carried out without numerical errors such as the Ising model [7–11], the dimer model [12–14], percolation model [15], the spanning tree model [16], and resistor network [17]. The exact study of the model subject to boundary conditions is of fundamental importance: (i) first, it represents new challenges for the unsolved lattice-statistical problems; (ii) second, it is crucial for the finite-size analysis; (iii) furthermore, it provides an optimal test bed for the predictions of the conformal field theory. Therefore, in recent decades there have many investigations on finite-size scaling, finite-size corrections and boundary effects for critical model systems [18–62]. Two-dimensional models of statistical mechanics have long served as a proving ground in attempts

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to understand critical behavior and to test the general ideas of FSS.

In this paper we are going to review the recent exact studies on finite size and boundary effects for the critical two-dimensional free-fermion models, including Ising model, dimer model, spanning tree model and resistor network model.

2 Finite size corrections and boundary effects in the critical two-dimensional free-fermion models

The exact asymptotic expansion of the free energy per site on an infinite cylinder (strip) of circumference L for exactly solvable models can easily be obtained by direct application of the Euler-Maclaurin summation formula [33]. However, derivation of such an expansion on a torus (cylinder or plane) of area S and aspect ratio ξ is much more difficult problem. For the Ising model on torus, such an expansion has first been studied by Ferdinand and Fisher [8]. Exploiting the exactly known partition function ($Z_{M,N}$) of the two dimensional Ising model on finite $M \times N$ square lattice with toroidal boundary conditions [9], they have calculated two leading terms, $f_{\rm bulk}$ and $f_0(\xi)$ in the expansion of the free energy per site $F_{T=Tc}(\xi, S) = -\frac{1}{MN} \ln Z_{M,N}$. In general, the exact asymptotic expansion of the free energy per site for the two-dimensional Ising model on $M \times N$ torus, cylinder or plane can be written as [23–25]

$$F_{T=Tc}(\xi, S) = f_{\rm bulk} + \frac{2Nf_{1s}}{S} + \frac{2Mf_{2s}}{S} + \frac{f_0(\xi)}{S} + \sum_{p=1}^{\infty} f_p(\xi)\; S^{-\frac{p}{2}-1} \tag{1}$$

where S is the area of the lattice, ξ is the aspect ratio

$$S = MN, \quad \xi = \frac{M}{N} \tag{2}$$

and f_{1s}, f_{2s} are the free energies per unit edge length in the horizontal and vertical directions respectively, which in the case of toroidal boundary conditions are equal to zero $f_{1s} = f_{2s} = 0$. In general, the bulk free energy $f_{\rm bulk}$, the surface free energies f_{1s} and f_{2s} and subleading correction terms $f_p(\xi)$ $(p = 1, 2, 3, \ldots)$ are non-universal, but the coefficient f_0 contains universal part $f_{\rm univ}$ and also non-universal, geometry-independent constant $f_{\rm nonuniv}$ $(f_0 = f_{\rm univ} + f_{\rm nonuniv})$ [63,64]. The universal part of f_0 depend only on the shape of the system and, possibly, the nature of the boundary conditions. In the case of the infinitely long strip the coefficient f_0 is universal and is known [18,65] to be related to the conformal anomaly number c and conformal weights of the underlying conformal theory. In the case of free boundary conditions on the square lattice Cardy and Peschel [66] have shown that corners on the boundary induce a trace anomaly in the stress tensor. They predicted that the contribution to the free energy from a corner with angle γ gives rise to a term in f_0 equal to

$$f_{\rm corner} = \frac{c}{48}\left(\frac{\gamma}{\pi} - \frac{\pi}{\gamma}\right) \ln S \tag{3}$$

where c is the central charge defining the universality class of the system and S is the area of the domain. For the Ising model on a finite triangular lattice on plane with free boundaries in five shapes: triangular, rhomboid, trapezoid, hexagonal and rectangular the corner parts of the free energy, internal energy, and specific heat has been calculated very accurately in [60,61]. In particular they calculate the corner free energy for angles $\gamma = \pi/3, \pi/2$ and $2\pi/3$ and find that they are in full agreement with conformal field theory prediction given by equation (3).

For the Ising model on a finite square lattice on plane with free boundaries in rectangular shapes the corner contribution to free energy $f_{\rm corner}$ comes from the corners with angle $\gamma = \pi/2$ and each corner gives rise to a term in f_0 equal to

$$f_{\rm corner} = -\frac{c}{32} \ln S, \tag{4}$$

later on, Imamura et al. study the corner terms with different boundary conditions within CFT [67]. According to their results, the contribution to the free energy from a corner with two edges under a and b conformally invariant boundary conditions is given by

$$f_{\rm corner} = -\left(\frac{c}{32} - \Delta_{a,b}\right) \ln S \tag{5}$$

where $\Delta_{a,b}$ is the conformal weight of the boundary operator inserted at the corner that changed boundary conditions from a to b. For the Ising model there are three different conformal weights, namely, $\Delta_{a,b} = 0, 1/16$ and $1/2$. There are also three different conformally invariant boundary conditions. Using bond propagation algorithms with surface fields the Ising model on a finite square lattice on plane in square shape with different boundary conditions has been studied in [62]. The exact results has been conjectured for the corner logarithmic term in the free energy, the internal energy, and the specific heat. The corner logarithmic terms in the free energy agree with the conformal field theory prediction.

Later, Kleban and Vassileva [64] extended the study of the free energy on a rectangle. They further derived a geometry-dependent universal part of the free energy in the rectangular geometry and showed that in addition to corner contribution predicted by Cardy and Peschel [66], the term f_0 contains also another universal part f_u depending on the aspect ratio

$$f_u = \frac{c}{4} \ln\left[\eta(q)\eta(q')\right], \tag{6}$$

where $\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ is the Dedekind eta function and $q = \exp(-2\pi\xi)$, $q' = \exp(-2\pi/\xi)$. Moreover, the term f_0 contains also non-universal, geometry-independent constant $f_{\rm nonuniv}$. Thus the term f_0 can be written as

$$f_0 = f_{\rm univ} + f_{\rm nonuniv}, \tag{7}$$

where the universal part f_{univ} of the free energy in the rectangular geometry can be calculated by conformal field theory methods [64] and given by

$$f_{\mathrm{univ}} = 4 f_{\mathrm{corner}} + f_u, \tag{8}$$

while non-universal part f_{nonuniv} of the free energy is not calculable via the conformal field theory methods. Equations (4)–(8) applies for any conformally invariant boundary condition around the edge of the rectangle. For the Ising model on plane the non-universal part f_{nonuniv} of the free energy has been calculated in [25]. In 2017 Baxter [51] and Hucht [52,53] has verified the CFT results for the Ising model in the rectangular geometry by an explicit calculations.

There has been much effort in understanding the behavior of finite-size corrections of the free energy, internal energy and specific heat. In 1999, Izmailian et al. [26] calculated finite-size corrections for cluster numbers of the q-state Potts model for q being $1, 2, 3$ and 4. In such studies, they extended the calculations of Ferdinand and Fisher for the Ising model on torus [8] to higher orders. In 2001 Lu and Wu [29] find finite-size corrections for free energy of the Ising model on an quadratic $N \times N$ lattice embedded on a Mobius strip and a Klein bottle to order N^{-1}. In 2002, Izmailian and Hu [44] extended the results of [8] for the free energy and the internal energy for the Ising model on torus up to order N^{-5} and for the specific heat up to order N^{-3}. In 2002, Janke and Kenna [27] has calculated the finite-size corrections of the specific heat for the Brascamp-Kunz boundary condition up to N^{-3} order. In 2002, Caselle et al. [68] used conformal field theory to study correction terms for the free energy and its derivatives of the Ising model. In the same year Ivashkevich et al. [23] provided a systematic method to compute finite-size corrections to the partition function and their derivatives of the Ising model on torus and derive *all* terms in this asymptotic expansion for the Ising model on finite square lattices with periodic boundary conditions. Their approach is based on an intimate relation between the terms of the asymptotic expansion and the so-called Kronecker's double series [23] which are directly related to elliptic theta functions. Using this approach, exact finite-size corrections for the critical Ising model [24,36,43,45], the dimer model [46–48,69], the spanning tree [70] and resistor network [71,72] on planar lattices with various boundary conditions have been obtained. Besides the aesthetic appeal of the exact expansion, there is also physical motivation to study non-universal corrections to FSS. The problem is that in numerical simulations of lattice models one usually studies relatively small lattices. Therefore, to compare the results of high precision numerical simulations to the theoretical predictions one cannot neglect sub-leading corrections to FSS [73].

The non-universal finite-size corrections can be calculated by the means of a perturbated conformal field theory [74,75]. In general, any lattice Hamiltonian (H) will contain correction terms to the critical Hamiltonian H_c

$$H = H_c + \sum_p g_p \int_{-N/2}^{N/2} \phi_p(v) dv,$$

where g_p is a non-universal constant and $\phi_p(v)$ is a perturbative conformal field. Then the eigenvalues of H are

$$E_n = E_{n,c} + \sum_p g_p \int_{-N/2}^{N/2} \phi_p(v) dv + \ldots,$$

where $E_{n,c}$ are the critical eigenvalues of H. The matrix element $\langle n|\phi_g(v)n\rangle$ can be computed in terms of the universal structure constants (C_{nln}) of the operator product expansion [74]: $\langle n|\phi_g(v)n\rangle = (2/N)x_g C_{nln}$, where x_g is the scaling dimension of the conformal field $\phi_g(v)$. Thus, non-universal terms in the asymptotic expansion also provide important information about the structure of irrelevant operators in conformal field theory [76]. Such studies inspire further research on finite-size corrections for the Ising and other exactly solvable models.

The asymptotic expansion for the dimer model has first been studied by Ferdinand [8]. Starting with the explicit expression for the partition function [12–14] he calculated finite-size corrections up to the first order, f_{bulk}, f_{1s}, f_{2s} and $f_0(\xi)$ of the expansion for the free energy of the dimer model on $M \times N$ square lattices with both free and toroidal boundary conditions for different parities of M and N. It has been shown that the coefficients f_{1s}, f_{2s} depend on the type of boundary conditions but independent on the parities of the number of the lattice sites, M and N, along the axes and the coefficients $f_0(\xi)$ in this expansion are sensitive to the boundary conditions and the parities of the number of the lattice sites. In 2003 Izmailian et al. [48] has shown that the asymptotic expansion for the free energy of the dimer model can be written in the form given by equation (1) and derive *all* terms in this asymptotic expansion for the dimer model on finite square lattices under five different boundary conditions, namely, free, cylindrical, toroidal, Mobius strip, and Klein bottle. They find that the aspect-ratio dependence of finite-size corrections $f_p(\xi)$, for $p = 0, 1, 2, \ldots$ are sensitive to boundary conditions and the parity of the number of lattice sites along the lattice axis. In references [46,47] has been found that the finite-size corrections for the dimer model on the infinitely long strip under free and periodic boundary conditions depend in a crucial way on the parity of the lattice sites in the finite directions and show that such unusual finite-size behavior can be fully explained in the framework of the $c = -2$ logarithmic conformal field theory. In the same year Izmailian and Kenna [69] have analyze the partition function of the dimer model on an $M \times N$ triangular lattice wrapped on a torus obtained by Fendley et al. [77]. From a finite-size analysis they have found that the dimer model on such a lattice can be described by a conformal field theory having a central charge $c = -2$.

Very recently [70], Izmailian and Kenna using the method of references [23,78] to derive exact finite-size corrections for the logarithm of the partition function of the spanning-tree model on the $M \times N$ square lattice with five different sets of boundary conditions in the form given by equation (1). Except for the bulk free energy f_{bulk}, all other coefficients in this expansion are sensitive to the boundary conditions. They also explain an apparent discrepancy between logarithmic correction terms in the free energy for a two-dimensional spanning-tree model with periodic and free-boundary conditions and conformal field theory predictions [79,80] and have obtained corner free energy for the spanning tree under free-boundary conditions in full agreement with conformal field theory predictions.

Using the exact expression for the resistance between arbitrary two nodes for finite $M \times N$ rectangular network obtained in reference [17] and the algorithm of reference [23], Izmailian and Huang [71] derive the exact asymptotic expansion for the resistance between two maximally separated nodes on an $M \times N$ rectangular network of resistors under free, periodic and cylindrical boundary conditions with resistors r and s in the two spatial directions. They show that the exact asymptotic expansion of the resistance between two maximally separated nodes of the network for all boundary conditions can be written as

$$\frac{1}{s}R_{M,N}(r,s) = c(\rho)\ln S + c_0(\rho,\xi) + \sum_{p=1}^{\infty} c_{2p}(\rho,\xi)\ S^{-p} \quad (9)$$

where S is the area of the lattice and ξ is the aspect ratio given by equation (2) and ρ is defined as

$$\rho = r/s. \quad (10)$$

All coefficients in this expansion depends on the boundary conditions. Recently, using the same approach Izmailian and Kenna [72] derive the asymptotic expansions for the resistance between the center node and a node on the boundary of the $M \times N$ cobweb network of resistors, which can be written in the form given by equation (9).

2.1 Ising models on the square lattice under different boundary conditions

In this section, we will consider Ising model under different boundary conditions, namely on torus with periodic and helical boundary conditions and on the cylinder with Branskamp-Kunz boundary conditions. Partition functions of all these models can be written in terms of the only object, namely, the partition function with twisted boundary conditions $Z_{\alpha,\beta}(M,N,\mu,d)$. We will derive the exact asymptotic expansion of the free energy, internal energy and specific heat for Ising model under different boundary conditions at the critical point $\mu = 0$.

The Ising model is defined on a lattice G of N_s sites, the ith site of the lattice for $1 \le i \le N_s$ is assigned a classical spin variable s_i, which has values ± 1. The spins interact according to the Hamiltonian

$$\beta H = -J \sum_{<ij>} s_i s_j \quad (11)$$

where J is exchange energy, the sum runs over the nearest neighbor pairs of spins, and $\beta = 1/k_B T$ is the inverse temperature. The partition function of the Ising model is given by the sum over all spin configurations on the $M \times N$ lattice

$$Z^{\text{Ising}}_{M,N}(J) = \sum_s e^{-\beta H(s)}. \quad (12)$$

It is convenient to set up another parameterizations of the interaction constant J in terms of the mass variable μ

$$\mu = \ln\sqrt{\sinh(2J)}. \quad (13)$$

Critical point corresponds to the massless case $\mu = \mu_c = 0$.

2.1.1 Ising model under helical boundary conditions

Let us start with the Ising model under helical boundary conditions. For the Ising model on the $M \times N$ square lattice with the helicity factor $d \equiv D/M$, the system has periodic boundary conditions in the N direction and helical (tilted) boundary conditions in the M direction such that the i-site in the first column is connected with the mod$(i+D,M)$th site in the N column of the lattice [81,82]. These boundary conditions permit an analytical approach to the determination of a number of thermodynamic quantities. The Ising model on torus under periodic boundary conditions can be considered as limiting case of the helical boundary conditions with $d = 0$.

An explicit expression for the partition function of the Ising model on $M \times N$ helical torus is given by [82]

$$Z^{\text{Ising}}_{M,N}(J) = \frac{1}{2}\left(\sqrt{2}\cosh J\right)^{2MN} \times \left\{ I_{\frac12,\frac12} + I_{\frac12,0} + I_{0,\frac12} - \text{sgn}\left(\frac{T-T_c}{T_c}\right) I_{0,0} \right\}, \quad (14)$$

$$I^2_{\alpha,\beta} = \prod_{m=1}^{M}\prod_{n=1}^{N}\left\{ a - b\cos\left[2\pi\left(\frac{m+\beta}{M} - d\frac{n+\alpha}{N}\right)\right] - b\cos\left[2\pi\frac{n+\alpha}{N}\right]\right\}, \quad (15)$$

where $a = \left(1+\tanh^2 J\right)^2$ and $b = 2\tanh J\left(1-\tanh^2 J\right)$. In addition, the function sgn(x) denotes the sign of the value x and T_c is the critical temperature of the bulk system ($\sinh 2J_c = 1$).

Then an explicit expression for the partition function of the Ising model on $M \times N$ helical torus can be rewritten as [45]

$$Z^{\text{Ising}}_{M,N}(\mu) = \frac{1}{2}\left(\sqrt{2}e^{\mu}\right)^{MN}\left\{ Z_{\frac12,\frac12}(M,N,\mu,d) + Z_{0,\frac12}(M,N,\mu,d) + Z_{\frac12,0}(M,N,\mu,d) + \text{sgn}(\mu) Z_{0,0}(M,N,\mu,d)\right\}, \quad (16)$$

where we have introduced the partition function with twisted boundary conditions $Z_{\alpha,\beta}(M,N,\mu,d)$

$$Z^2_{\alpha,\beta}(M,N,\mu,d) = \prod_{n=0}^{N-1}\prod_{m=0}^{M-1} 4\left\{ \sin^2\left[\frac{\pi}{N}(n+\alpha)\right] + \sin^2\left[\frac{\pi}{M}\left(m+\beta-d\xi(n+\alpha)\right)\right] + 2\sinh^2\mu\right\}, \quad (17)$$

where ξ is the aspect ratio given by equation (2). Here α control the boundary conditions for the underlying free fermion in the N-direction ($\psi(x+N,y)=e^{2\pi i\alpha}\psi(x,y)$), while β control the boundary conditions for the underlying free fermion in the M-direction ($\psi(x,y+M)=e^{2\pi i\beta}\psi(x,y)$). In particular $\alpha=0$ or $\beta=0$ corresponds to the periodic boundary conditions in the N or M directions respectively, while $\alpha=\frac{1}{2}$ or $\beta=\frac{1}{2}$ stands for anti-periodic boundary conditions. In general the twist angles α and β can be taken from the interval $[0,1]$ [23]. The $Z_{\alpha,\beta}(M,N,\mu,d)$ can be transformed into a simpler form

$$Z_{\alpha,\beta}(M,N,\mu,d)=\prod_{n=0}^{N-1}2\Big|\sinh\Big\{M\omega_\mu\left(\pi\frac{n+\alpha}{N}\right)+i\pi\left[\beta-d\xi(n+\alpha)\right]\Big\}\Big|, \tag{18}$$

where lattice dispersion relation has appeared

$$\omega_\mu(k)=\sqrt{\sin^2k+2\sinh^2\mu}. \tag{19}$$

This is nothing but the functional relation between energy $\omega_\mu(k)$ and momentum k of a free quasi-particle on the planar square lattice. In what follow we will need the Taylor expansion of lattice dispersion relation at the critical point $\omega_0(k)$

$$\omega_0(k)=k\left(\lambda+\sum_{p=1}^{\infty}\frac{\lambda_{2p}}{(2p)!}\,k^{2p}\right) \tag{20}$$

where $\lambda=1$, $\lambda_2=-2/3$, $\lambda_4=4$, etc. We shall not use the special values of these coefficients assuming the possibility for generalizations.

It is easy to see from equation (17), that the partition function with twisted boundary conditions $Z_{\alpha,\beta}(M,N,\mu,d)$ obey the following very useful identities for the particular case $d=0$ and $\mu=0$ (see [48])

$$Z_{\alpha,\beta}(M,2N,0,0)=Z_{\frac{\alpha}{2},\beta}(M,N,0,0)Z_{\frac{1-\alpha}{2},\beta}(M,N,0,0), \tag{21}$$

$$Z_{\alpha,\beta}(2M,N,0,0)=Z_{\alpha,\frac{\beta}{2}}(M,N,0,0)Z_{\alpha,\frac{1-\beta}{2}}(M,N,0,0). \tag{22}$$

In particular, from the identities given by equations (21) and (22) one can obtain that

$$Z_{\frac{1}{2},0}(M,2N,0,0)=Z^2_{\frac{1}{4},0}(M,N,0,0), \tag{23}$$

$$Z_{\frac{1}{2},\frac{1}{2}}(M,2N,0,0)=Z^2_{\frac{1}{4},\frac{1}{2}}(M,N,0,0), \tag{24}$$

$$Z_{\frac{1}{4},0}(2M,N,0,0)=Z_{\frac{1}{4},0}(M,N,0,0)Z_{\frac{1}{4},\frac{1}{2}}(M,N,0,0), \tag{25}$$

$$Z_{\frac{1}{2},0}(2M,N,0,0)=Z_{\frac{1}{2},0}(M,N,0,0)Z_{\frac{1}{2},\frac{1}{2}}(M,N,0,0). \tag{26}$$

We are interested in computing the asymptotic expansions of the free energy F at the critical point $J=J_c$ ($\mu=\mu_c=0$) for large N, M and D with fixed aspect ratio ρ and helicity factor d. The free energy F is defined as follows

$$F=-\frac{1}{S}\ln Z^{\rm Ising}_{M,N}(J) \tag{27}$$

and can be compute directly from equation (16).

Note that the general theory about the asymptotic expansion of $Z_{\alpha,\beta}(M,N,0,d)$, for the particular case $d=0$, has been given in [23]. The theory can be extended to the case with arbitrary rational number d [45]. The exact asymptotic expansion of the $\ln Z_{\alpha,\beta}(M,N,0,d)$ in terms of the Kronecker's double series has been obtained in [45]

$$\ln Z_{\alpha,\beta}(M,N,0,d)=\frac{2G}{\pi}S+\ln\left|\frac{\theta_{\alpha,\beta}(i\tau_0\xi)}{\eta(i\tau_0\xi)}\right| -2\pi\xi\sum_{p=1}^{\infty}\left(\frac{\pi^2\xi}{S}\right)^p\frac{\mathrm{Re}\;\Lambda_{2p}\,K^{\alpha,\beta}_{2p+2}(i\tau_0\xi)}{(2p+2)(2p)!}. \tag{28}$$

Here $G=0.915966$ is Catalans constant and the differential operators Λ_{2p} can be expressed via coefficients λ_{2p} of the expansion of the lattice dispersion relation given by equation (20) as

$$\begin{aligned}
\Lambda_2&=\lambda_2\\
\Lambda_4&=\lambda_4+3\lambda_2^2\,\frac{\partial}{\partial\tau_0}\\
\Lambda_6&=\lambda_6+15\lambda_4\lambda_2\,\frac{\partial}{\partial\tau_0}+15\lambda_2^3\,\frac{\partial^2}{\partial\tau_0^2}\\
&\vdots\\
\Lambda_p&=\sum_{r=1}^{p}\sum\left(\frac{z_{p_1}}{p_1!}\right)^{k_1}\ldots\left(\frac{z_{p_r}}{p_r!}\right)^{k_r}\frac{p!}{k_1!\ldots k_r!}\,\frac{\partial^k}{\partial z^k}
\end{aligned} \tag{29}$$

here summation is over all positive numbers $\{k_1,\ldots,k_r\}$ and different positive numbers $\{p_1,\ldots,p_r\}$ such that $p_1k_1+\cdots+p_rk_r=p$ and $k=k_1+\cdots+k_r-1$.

Thus we can see that the asymptotic expansion of $Z_{\alpha,\beta}(M,N,0,d)$ can be expressed in terms of the analytic functions such as the elliptic theta functions $\theta_{\alpha,\beta}(\tau)$, Dedekind η-function and Kronecker's double series, $K_p^{\alpha,\beta}(\tau)$ with $\tau=i\tau_0\xi$ (see [23,45]), where τ_0 is defined as

$$\tau_0=\lambda-id. \tag{30}$$

Note that Kronecker's double series, $K_p^{\alpha,\beta}(\tau)$ can be expressed in terms of the elliptic theta functions only [23,45].

Equations (16), (27) and (28) implies that the free energy at the critical point ($\mu=\mu_c=0$) can be written in the form given by equation (1). Thus, the finite-size corrections to the free energy are always integer powers of S^{-1}. The first few coefficients in the exact asymptotic expansion of the free energy are given by

$$f_{\rm bulk} = -\ln\sqrt{2} - \frac{2G}{\pi}, \quad (31)$$

$$f_{1s} = f_{2s} = 0, \quad (32)$$

$$f_0(\xi) = -\ln\frac{\theta_2+\theta_3+\theta_4}{2\eta}, \quad (33)$$

$$f_1(\xi) = -\frac{\pi^3\xi^2}{180}\frac{\frac{7}{8}(\theta_2^9+\theta_3^9+\theta_4^9)+\theta_2\theta_3\theta_4\left[\theta_2^3\theta_4^3-\theta_3^3\theta_2^3-\theta_3^3\theta_4^3\right]}{\theta_2+\theta_3+\theta_4}, \quad (34)$$

$$\vdots$$

To simplify the notation we have use the short-hands

$$\theta_k = |\theta_k(i\tau_0\xi)|, \quad k=2,3,4, \qquad \eta = |\eta(i\tau_0\xi)|. \quad (35)$$

All finite size correction terms are invariant under transformation $\xi \to 1/(1+d^2)\xi$, which actually means that $\xi_{\rm eff}$

$$\xi_{\rm eff} = \xi\sqrt{1+d^2} \quad (36)$$

can be regarded as the effective aspect ratio.

2.1.2 Ising model on torus

An explicit expression for the partition function of the Ising model on $M \times N$ torus, which was given originally by Kaufmann [9], can be written as

$$Z^{\rm Ising}_{M,N}(\mu) = \frac{1}{2}\left(\sqrt{2}e^{\mu}\right)^{MN}\Big\{Z_{\frac{1}{2},\frac{1}{2}}(M,N,\mu,0) + Z_{0,\frac{1}{2}}(M,N,\mu,0) + Z_{\frac{1}{2},0}(M,N,\mu,0) + Z_{0,0}(M,N,\mu,0)\Big\} \quad (37)$$

where the partition function with twisted boundary conditions $Z_{\alpha,\beta}(M,N,\mu,0)$ are given by equation (17) with $d=0$. To derive the exact asymptotic expansion of the free energy for Ising model at the critical point $\mu=0$ one need to derive the exact asymptotic expansion of the logarithm of the partition function with twisted boundary conditions $Z_{\alpha,\beta}(M,N,0,0)$. Exact asymptotic expansion of the $Z_{\alpha,\beta}(M,N,0,0)$ have been obtained in 2002 by Ivashkevich et al. [23].

The exact asymptotic expansion of $Z_{\alpha,\beta}(M,N,0,0)$ are given by equations (28) for the case $d=0$. The first few coefficients in the exact asymptotic expansion of the free energy are given by equations (31)–(34) for the case $d=0$.

2.1.3 Ising model under Brascamp-Kunz boundary conditions

As is mentioned in introduction there are a few boundary conditions for which the Ising model has been solved exactly. Among them is the special boundary conditions studied by Brascamp and Kunz (BK) [7] on $M\times 2N$ square lattice. They considered a lattice with $2N$ sites in the x direction and M sites in the y directions. The boundary conditions are periodic in the x direction; in the y directions, the spins are up (+1) along the upper border of the resulting cylinder and have the alternating values along the lower border of the resulting cylinder. For the BK boundary conditions, the Ising partition function given in reference [7] can be rewritten as

$$Z^{\rm Ising}_{M,2N}(\mu) = (\sqrt{2}e^{\mu})^{2MN}\prod_{i=1}^{N}\prod_{j=1}^{M}F(i,j) \quad (38)$$

where $\mu = 1/2\ln\sinh 2J$ and

$$F(i,j) = 4\left[2\sinh^2\mu + \sin^2\left(\frac{\pi(i-1/2)}{2N}\right) + \sin^2\left(\frac{\pi j}{2(M+1)}\right)\right]. \quad (39)$$

It can be shown [24] that the partition function $Z^{\rm Ising}_{M,2N}$ can be expressed to the form of partition function with twisted boundary conditions $Z_{1/2,0}(M,N,\mu,0)$ as

$$Z^{\rm Ising}_{M,2N}(\mu) = \frac{\left(\sqrt{2}e^{\mu}\right)^{2MN}}{2\sqrt{\cosh\left[2N\omega_\mu(0)\right]\cosh\left[2N\omega_\mu(\pi/2)\right]}} \times\sqrt{Z_{1/2,0}(2M+2,2N,\mu,0)}, \quad (40)$$

where $Z_{1/2,0}(2M+2,2N,\mu,0)$ is given by equation (18) with $d=0$, $\alpha=1/2$ and $\beta=0$ and $\omega_\mu(k)$ is given by equation (19).

Now using equation (28) with $d=0$, $\alpha=1/2$ and $\beta=0$ one can easily obtain the exact asymptotic expansion of the free energy for Ising model with Brascamp-Kunz boundary conditions, which can be written in the form given by equation (1). The bulk free energy is the same for all boundary conditions and given by equation (31). All other coefficients in that expansion are given by

$$f_{1s} = \frac{1}{4}\ln 2(1+\sqrt{2}), \quad f_{2s}=0, \quad (41)$$

$$f_0 = -\frac{1}{2}\ln\frac{\theta_4(i\xi)}{2\eta(i\xi)}, \quad (42)$$

$$f_p = \frac{\pi^{2p+1}\xi^{p+1}}{2^p(2p+2)(2p)!}\,\Lambda_{2p}\,K^{\frac{1}{2},0}_{2p+2}(i\xi) \ \text{ for } \ p=1,2,3,\ldots, \quad (43)$$

where ξ is given by equation (2) with M replaced by $M+1$, namely

$$\xi = \frac{M+1}{N} \quad (44)$$

and S is given by

$$S = 2(M+1)N. \quad (45)$$

Note that S is slightly different from the true area of the $M\times 2N$ square lattice and aspect ratio is twice larger than true aspect ratio of the lattice. The coefficients Λ_{2p} are listed in equation (29) and Kronecker's double series $K_{2p+2}^{\frac{1}{2},0}$ can be written in terms of the elliptic theta functions for $p = 1, 2, 3, \ldots$ (see Appendix F of [23]). Thus all coefficients in the expansion equation (1) are expressed through analytical functions.

It is interesting to mention that the internal energy at the critical point (U_c) of the Ising model with Brascamp and Kunz boundary condition on finite $M \times 2N$ square lattice is equal to its bulk values without any finite-size corrections, namely $U_c = -\sqrt{2}$. To see that let us consider the internal energy at the critical point $T = T_c$ ($\mu = 0$)

$$U_c = \left.\frac{dF(J)}{dJ}\right|_{J=J_c} = \\ -\mu'\left(1 + \frac{d}{d\mu}\ln Z_{1/2,0}(2M+2, 2N, \mu, 0)|_{\mu=0}\right) = -\sqrt{2}, \quad (46)$$

where $\mu' = \left.\frac{d\mu}{dJ}\right|_{J=J_c}$ is the first derivative of $\mu(J)$ with respect to J at criticality. The derivatives μ' can be easily computed from equation (13): $\mu' = \sqrt{2}$. One can further note that $Z_{1/2,0}(2M+2, 2N, \mu, 0)$ is an even function with respect to its argument μ, which imply immediately that $\left(dZ_{1/2,0}(2M+2, 2N, \mu, 0)/d\mu\right)_{\mu=0} = 0$. Thus we find that $U_c = -\sqrt{2}$.

2.2 The dimer model on square lattice under different boundary conditions

The dimer model was originally introduced to represent physical adsorption of diatomic molecules on crystal surfaces [83]. The surface may be considered as a regular lattice which attracts the diatomic molecules (dimers) in such a way that each dimer fills two neighboring lattice sites and with crucial constraint that no lattice site is covered by two dimers. In contrast to spin models, the critical behavior of dimer models are strongly influenced by the structure of the underlying lattice. For example, the square lattice dimer model is critical with algebraic decay of correlators [84,85], while the dimer model on the anisotropic honeycomb lattice, which is equivalent to a five-vertex model on the square lattice [86], exhibits a potassium dihydrogen phosphate (KDP)-type singularity and the dimer model on the Fisher-type lattice exhibits Ising-type transitions [87]. Thus, it appears that the dimer model itself has not a single critical behavior, but several critical behaviors associated with different classes of universality.

The interest in dimer model was renewed with the discovery of high-temperature superconductivity and also with recent work on domino tilings (which are equivalent to dimers on a square lattice) of an Aztec diamond, demonstrating a strong effect of the boundary on a typical domino configuration [77,88–90].

The exact calculation of partition functions of the dimer model on the $M \times N$ square lattice under different boundary conditions has attracted the attention of researchers for more than fifty years. In 1961 Kasteleyn [12] obtained exact partition functions for the dimer model on the square lattice with both free and toroidal boundary conditions. Fisher [13], Temperley and Fisher [14] also solved the case of free boundary case independently. Ferdinand [91] calculated finite-size corrections up to the first order for the free energy of the dimer model on $M \times N$ square lattices with both free and toroidal boundary conditions for different parities of M and N. In 1973 McCoy and Wu [10] calculated exact partition functions for cylindrical boundary conditions. In 1985 Bhattacharjee and Nagle [92] studied the finite-size effect of an anisotropic dimer model of domain walls on the brick lattice. In 1993 Brankov and Priezzev [93] obtained the exact partition function for a Möbius strip. In 1999 and 2002 Lu and Wu obtained exact partition functions for a Möbius strip and a Klein bottle [94,95] and calculated finite-size corrections up to the first order for $M \times N$ lattices when both M and N are even. In 2011, Wu et al. [96] solve the monomer-dimer problem on a nonbipartite lattice, a simple quartic lattice with cylindrical boundary conditions, with a single monomer residing on the boundary by mapping the problem onto one of closed-packed dimers on a related lattice. In 2014, Allegra and Fortin [97] apply the Grassmann algebra to the monomer-dimer problem on a square lattice. In 2015, Allegra [98] perform the finite size effect analysis to study surface and corner effects of the dimer model on the square lattice with an arbitrary number of monomers.

In this section, we will consider the model on the square lattice under five different boundary conditions. We will relate the exact partition functions of the dimer model on the square lattice under free, cylindrical, toroidal, Möbius strip and Klein bottle boundary conditions obtained by Kasteleyn [12], Temperley and Fisher [13,14], McCoy and Wu [10], Brankov and Priezzhev [93], and Lu and Wu [94,95] to the partition functions with twisted boundary conditions $Z_{\alpha,\beta}(M, N, 0, 0)$ with $(\alpha, \beta) = (1/2, 0)$, $(0, 1/2)$ and $(1/2, 1/2)$. Then we can apply the algorithm of reference [23] to derive the exact asymptotic expansions of the logarithm of the partition functions for all boundary conditions. We find that the finite-size corrections is sensitive to boundary conditions and the parity of the number of lattice sites along the lattice axis.

Consider a dimer model on an $M \times N$ square lattice of MN sites with M rows and N columns. The lattice forms a cylinder if there are periodic boundary conditions in the horizontal directions and free boundary conditions in the vertical direction, a torus if there are periodic boundary conditions in both directions, a Möbius strip if there are twisted boundary conditions in the horizontal direction and free boundary conditions in the vertical direction, and a Klein bottle if, in addition to the twisted boundary conditions in the horizontal directions, there are periodic boundary conditions in the vertical direction. Here under twisted boundary condition we mean the twisting one end of a rectangular strip through 180° about the horizontal axis of the strip and attaching this end to the other.

The partition function of the dimer model on an $M \times N$ lattice is given by

$$Z_{M,N} = \sum z_v^{n_v} z_h^{n_h}, \tag{47}$$

where summation is taken over all dimer covering configurations, z_v and z_h are, respectively, dimer weight in the horizontal and vertical directions, n_v and n_h are, respectively, the number of vertical and horizontal dimers. Without lose the generality we will consider the case $z = 1$ ($z_h = z_v$). We will show that the exact asymptotic expansion of the logarithm of the partition function for the dimer model can be written in the form given by equation (1).

The explicit expression of the partition function depends crucially on whether M and N being even or odd, and since the total number of sites must be even if the lattice is to be completely covered by dimers, we will consider three cases, namely ($M \to 2M$, $N \to 2N$), ($M \to 2M-1$, $N \to 2N$), and ($M \to 2M$, $N \to 2N-1$). It has been shown [23] that the partition function for the dimer model under five different boundary conditions can be expressed in terms of the only object $Z_{\alpha,\beta}(M, N, 0, 0)$ with $(\alpha, \beta) = (0, \frac{1}{2}), (\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2})$.

(i) For the partition function of the dimer model on $2M \times 2N$ lattice under different boundary conditions we obtained [23]

$$Z^{torus}_{2M,2N} = \frac{z_v^{2MN}}{2}\Big[Z^2_{\frac{1}{2},\frac{1}{2}}(M,N,0,0) + Z^2_{0,\frac{1}{2}}(M,N,0,0) + Z^2_{\frac{1}{2},0}(M,N,0,0)\Big], \tag{48}$$

$$Z^{free}_{2M,2N} = z_v^{2MN}\left[\frac{\sqrt{2}\,Z_{\frac{1}{2},\frac{1}{2}}(2M+1,2N+1,0,0)}{2\cosh\left[(2M+1)a\right]\cosh\left[(2N+1)a\right]}\right]^{\frac{1}{2}}, \tag{49}$$

$$Z^{cyl}_{2M,2N} = z_v^{2MN}\frac{Z_{\frac{1}{2},\frac{1}{2}}(2M+1,N,0,0)}{2\cosh(Na)}, \tag{50}$$

$$Z^{Klein}_{2M,2N} = z_v^{2MN} Z_{\frac{1}{2},\frac{1}{2}}(M,2N,0,0), \tag{51}$$

$$Z^{Mob}_{2M,2N} = z_v^{2MN}\left[\frac{Z_{\frac{1}{2},\frac{1}{2}}(2M+1,2N,0,0)}{2\cosh(2Na)}\right]^{\frac{1}{2}}, \tag{52}$$

where the partition function with twisted boundary conditions $Z_{\alpha,\beta}(M, N, 0, 0)$ is given by equation (18) with $\mu = 0$ and $d = 0$. Here the constant a is defined as follow

$$a = \operatorname{arcsinh} 1.$$

Note that the general theory about the asymptotic expansion of $Z_{\alpha,\beta}(M, N, 0, 0)$ has been given in [23].

(ii) For the partition function of the dimer model on $(2M-1) \times 2N$ lattice under different boundary conditions we obtained [23]

$$Z^{torus}_{2M-1,2N} = z_v^{N(2M-1)} Z_{\frac{1}{2},0}(2M-1,N,0,0), \tag{53}$$

$$Z^{free}_{2M-1,2N} = z_v^{N(2M-1)}\left[\frac{Z_{\frac{1}{2},0}(2M,2N+1,0,0)}{\sqrt{2}\sinh(2Ma)\cosh((2N+1)\,a)}\right]^{\frac{1}{2}}, \tag{54}$$

$$Z^{cyl}_{2M-1,2N} = \frac{z_v^{N(2M-1)}}{2}\frac{Z_{\frac{1}{2},0}(2M,N,0,0)}{\cosh(N\,a)}, \tag{55}$$

$$Z^{Klein}_{2M-1,2N} = \sqrt{2} z_v^{N(2M-1)} Z^{\frac{1}{2}}_{\frac{1}{2},0}(2M-1,2N,0,0), \tag{56}$$

$$Z^{Mob}_{2M-1,2N} = z_v^{N(2M-1)}\left[\frac{Z_{\frac{1}{2},0}(2M,2N,0,0)}{\cosh(2N\,a)}\right]^{\frac{1}{2}}. \tag{57}$$

(iii) And for the partition function of the dimer model on $2M \times (2N-1)$ lattice under different boundary conditions we get [23]

$$Z^{torus}_{2M,2N-1} = z_v^{M(2N-1)} Z_{0,\frac{1}{2}}(M,2N-1,0,0), \tag{58}$$

$$Z^{free}_{2M,2N-1} = z_v^{M(2N-1)}\left[\frac{\sqrt{2}Z_{0,\frac{1}{2}}(2M+1,2N,0,0)}{\sqrt{2}\sinh(2N\,a)\cosh((2M+1)\,a)}\right]^{\frac{1}{2}}, \tag{59}$$

$$Z^{cyl}_{2M,2N-1} = z_v^{M(2N-1)}\left[\frac{Z_{0,\frac{1}{2}}(2M+1,2N-1,0,0)}{2\sinh((2N-1)\,a)}\right]^{\frac{1}{2}}, \tag{60}$$

$$Z^{Klein}_{4M,2N-1} = z_v^{2M(2N-1)} Z_{\frac{1}{2},\frac{1}{2}}(2M,2N-1,0,0), \tag{61}$$

$$Z^{Mob}_{4M,2N-1} = z_v^{2M(2N-1)}\left[\frac{Z_{\frac{1}{2},\frac{1}{2}}(4M+1,2N-1,0,0)}{2\cosh((2N-1)\,a)}\right]^{\frac{1}{2}}. \tag{62}$$

2.2.1 Identities of the dimer model on the square lattice

Now using equations (49)–(62) it is easy to derive a whole set of the identities for the dimer model with different boundary conditions. From equations (50) and (52) its easy to see that dimer model on $2M \times 2N$ lattice with cylindrical and Möbius strip boundary conditions obey the following identity

$$Z^{Mob}_{2M,2N} = \sqrt{Z^{cyl}_{2M,4N}}. \tag{63}$$

This identity was first established in the large M and N limit by Brankov and Priezzhev [93] and then was rigorously established by Lu and Wu [95]. Now that identity follows directly from the definition of the partition function of the dimer model through the partition function with twisted boundary conditions $Z_{\frac{1}{2},\frac{1}{2}}(M, N, 0, 0)$ given by equations (50) and (52). Now it is easy to derive another two identities for the partition functions of the dimer model on $2M \times 2N$ lattices, namely, from equations (49) and (50) its easy to see that dimer model on $2M \times 2N$

lattice with free and cylindrical boundary conditions obey the following identity

$$Z_{2M,2N}^{free} = \frac{2^{\frac{1}{4}}}{z_v^M\sqrt{\cosh\left[(2M+1)\,a\right]}}\sqrt{Z_{2M,4N+2}^{cyl}} \qquad (64)$$

and from equations (50) and (51) its easy to see that dimer model on $2M\times 2N$ lattice with cylindrical and Klein bottle boundary conditions obey the following identity

$$Z_{4M+2,2N}^{Klein} = 2z_v^{-2N}\cosh\left(2Na\right)Z_{2M,4N}^{cyl}. \qquad (65)$$

The identities given by equations (64) and (65) was first established by Izmailian et al. [48].

Now let us consider the dimer model on the $(2M-1)\times 2N$ lattice. Using equations (53) and (56) one can write the following identities between partition functions of the dimer model on $(2M-1)\times 2N$ lattice with toroidal and Klein bottle boundary conditions

$$Z_{2M-1,4N}^{torus} = \frac{1}{2}\left[Z_{2M-1,2N}^{Klein}\right]^2. \qquad (66)$$

Now from equations (55) and (57) its easy to see that dimer model on $(2M-1)\times 2N$ lattice with cylindrical and Möbius strip boundary conditions obey the following identity

$$Z_{2M-1,4N}^{cyl} = \frac{1}{2}\left[Z_{2M-1,2N}^{Mob}\right]^2, \qquad (67)$$

and from equations (55) and (57) we arrive to the following identity for the partition functions of the dimer model on $(2M-1)\times 2N$ lattice with cylindrical and free boundary conditions

$$Z_{2M-1,4N+2}^{cyl} = z_v^{2M-1}\frac{\sinh\left(2Ma\right)}{\sqrt{2}}\left[Z_{2M-1,2N}^{free}\right]^2. \qquad (68)$$

The identity given by equation (67) was first established by Lu and Wu [95] and identities given by equations (66) and (68) was first established by Izmailian et al. [48].

And finally, using the expressions for the partition functions $Z_{2M-1,2N}^{cyl}, Z_{2M-1,2N}^{Mob}$ and $Z_{2M,2N}^{Klein}$ (see Eqs. (55), (57) and (51)) and identity given by equation (26) we can obtain the following identities

$$Z_{4M-1,4N}^{cyl} = Z_{2M-1,4N}^{cyl}Z_{4M,2N}^{Klein} \qquad (69)$$

$$Z_{4M-1,2N}^{Mob} = Z_{2M-1,2N}^{Mob}\sqrt{Z_{4M,2N}^{Klein}}. \qquad (70)$$

The identities given by equations (69) and (70) was established by Izmailian et al. [48]. Note that there are few misprints in equations (70) and (71) of reference [48]. In equation (70) one should replace N by $2N$ in the expressions for Z^{cyl} and in equation (71) one should replace N by $N/2$ in the expression for Z^{Klein}.

Let us point out once again that the identities given by equations (63)–(70) were obtained by comparing exact expressions for the various partition functions given by equations (48)–(62). In 2016, Cimasoni and Pham [99] provide a general principle allow us, not only to explain some of these identities (Eqs. (63), (66) and (67)), but also to generalize them. There is still interest in explaining remaining identities (Eqs. (64), (65), (68)–(70)) by some general underlying principle.

2.2.2 Exact finite-size corrections of the free energy for square lattice dimer model under different boundary conditions

In Section 2 it has been shown that the partition functions of the dimer model with various boundary conditions can be expressed, in terms of the partition function with twisted boundary conditions $Z_{1/2,0}(K,L,0,0)$, $Z_{0,1/2}(K,L,0,0)$ and $Z_{1/2,1/2}(K,L,0,0)$ (see Eqs. (48)–(62)). Based on such results and using asymptotic expansion of the partition function with twisted boundary conditions given by equation (28) for $\mu=0$ and $d=0$, one can easily write down all the terms of the exact asymptotic expansion of the logarithm of the partition functions for the dimer model, which can be written in the form given by equation (1).

The bulk free energy f_{bulk} is the same for all boundary conditions and given by

$$f_{bulk} = -\frac{1}{2}\ln z_v - \frac{G}{\pi}, \qquad (71)$$

where G is the Catalan constant given by $G=\sum_{n=0}^{\infty}(-1)^n/(2n+1)^2 = 0.915965594\ldots$ The surface free energy f_{1s} and f_{2s} are given by

$$f_{1s}^{torus} = f_{1s}^{Klein} = 0,$$

$$f_{2s}^{torus} = f_{2s}^{Klein} = f_{2s}^{cyl} = f_{2s}^{Mob} = 0,$$

$$f_{2s}^{free} = f_{1s}^{free} = f_{1s}^{cyl} = f_{1s}^{Mob} = \frac{1}{4}\ln z_v + \frac{1}{4}\ln\left(1+\sqrt{2}\right). \qquad (72)$$

Note that f_{1s} and f_{2s} depend on the type of boundary conditions but independent on the parities (even or odd) of M and N. This is not the case for the other coefficients $f_p(\xi)$ $(p=0,1,\ldots)$ in the expansion of equation (1). Now, we will list expansion coefficients $f_p(\xi)$ for $p=0,1,2,\ldots$ and show that they depend crucially on whether M and N are even or odd.

(i) *Dimers on* $2M\times 2N$ *lattice.* Now using equation (28) with $d=0$, $(\alpha,\beta)=(0,1/2),(1.2,0)$ and $(1/2,1/2)$ and the expressions for the partition functions $Z_{2M,2N}^{torus}$ and $Z_{2M,2N}^{Mob}$ one can easily obtain the coefficients $f_p(\xi)$ $(p=0,1,\ldots)$ in the exact asymptotic expansion of the free energy for the dimer model on $2M\times 2N$ lattice under periodic (torus) and Möbius strip boundary conditions

$$f_0^{torus}(\xi) = -\ln\frac{\theta_2^2+\theta_3^2+\theta_4^2}{2\eta^2}, \quad (73)$$

$$f_1^{torus}(\xi) = -\frac{\pi^3\xi^2}{90}\,\frac{\frac{7}{8}\left(\theta_2^{10}+\theta_3^{10}+\theta_4^{10}\right)+\theta_2^2\theta_3^2\theta_4^2\left(\theta_2^2\theta_4^2-\theta_2^2\theta_3^2-\theta_3^2\theta_4^2\right)}{\theta_2^2+\theta_3^2+\theta_4^2}, \quad (74)$$

$$\vdots$$

$$f_0^{Mob}(\xi) = -\frac{1}{6}\ln\frac{2\,\theta_3^2}{\theta_2\theta_4}, \quad (75)$$

$$f_1^{Mob}(\xi) = -\frac{\pi^3\xi^2}{360}\left(\frac{7}{8}\theta_3^8+\theta_2^4\theta_4^4\right), \quad (76)$$

$$f_p^{Mob}(\xi) = \frac{\pi^{2p+1}\xi^{p+1}}{(2p+2)(2p)!}\Lambda_{2p}\,K_{2p+2}^{\frac{1}{2},\frac{1}{2}}(i\xi) \text{ for } p=2,3,\ldots, \quad (77)$$

where $\theta_i = \theta_i(i\xi)$ with $i = 2, 3, 4$. The coefficients Λ_{2p} are listed in equation (29) and Kronecker's double series $K_{2p+2}^{\frac{1}{2},\frac{1}{2}}(i\xi)$ can be written in terms of the elliptic theta functions for $p = 1, 2, 3, \ldots$ (see Appendix F of [23]).

Using the expressions for the partition functions $Z_{2M,2N}^{free}, Z_{2M,2N}^{cyl}, Z_{2M,2N}^{Klein}$ and $Z_{2M,2N}^{Mob}$ given by equations (49)–(52) one can easily obtain the expansion coefficients $f_p(\xi)$ $(p = 0, 1, \ldots)$ for free, cylindrical and Klein bottle boundary conditions through the following functional relations

$$f_p^{free}(\xi) = f_p^{Mob}(\xi) + \frac{\delta_{p,0}}{4}\ln(8z_v^2), \quad (78)$$

$$f_p^{cyl}(\xi) = 2^{p+1}\,f_p^{Mob}(2\xi), \quad (79)$$

$$f_p^{Klein}(\xi) = 2^{p+1}\,f_p^{Mob}(\xi/2). \quad (80)$$

(ii) *Dimers on* $(2M-1)\times 2N$ *lattice*. For the Möbius strip boundary condition the asymptotic expansion for the free energy of the dimer model on $(2M-1)\times 2N$ lattice can be obtain by using expressions for the partition functions $Z_{2M-1,2N}^{Mob}$ and the asymptotic expansion of $\ln Z_{\alpha,\beta}(M,N,0,0)$ given by equation (28) with $d = 0$, $\alpha = 1/2$ and $\beta = 0$. The first few coefficient in this expansion are given by

$$f_0^{Mob}(\xi) = -\frac{1}{2}\ln 2 - \frac{1}{6}\ln\frac{2\,\theta_4^2}{\theta_2\theta_3}, \quad (81)$$

$$f_1^{Mob}(\xi) = -\frac{\pi^3\xi^2}{360}\left(\frac{7}{8}\theta_4^8-\theta_2^4\theta_3^4\right), \quad (82)$$

$$f_p^{Mob}(\xi) = \frac{\pi^{2p+1}\xi^{p+1}}{(2p+2)(2p)!}\Lambda_{2p}\,K_{2p+2}^{\frac{1}{2},0}(i\xi) \quad \text{for} \quad p=2,3,\ldots. \quad (83)$$

Using the expressions for the partition functions $Z_{2M-1,2N}^{free}, Z_{2M-1,2N}^{cyl}, Z_{2M-1,2N}^{Klein}$ and $Z_{2M-1,2N}^{Mob}$ given by equations (53)–(57) one can easily obtain the expansion coefficients $f_p(\xi)$ $(p = 0, 1, \ldots)$ for periodic, free, cylindrical and Klein bottle boundary conditions through the following functional relations

$$f_p^{torus}(\xi) = 2^{p+1}\,f_p^{Mob}(2\xi) - \frac{\delta_{p,0}}{2}\ln 2, \quad (84)$$

$$f_p^{free}(\xi) = f_p^{Mob}(\xi) + \frac{\delta_{p,0}}{4}\ln(2z_v^2), \quad (85)$$

$$f_p^{cyl}(\xi) = 2^{p+1}\,f_p^{Mob}(2\xi) - \frac{\delta_{p,0}}{2}\ln 2, \quad (86)$$

$$f_p^{Klein}(\xi) = f_p^{Mob}(\xi). \quad (87)$$

(iii) *Dimers on* $2M\times(2N-1)$ *lattice*. Finally let us consider the dimer model on the $2M\times(2N-1)$ square lattice. For cylindrical and Möbius strip boundary conditions, the expansion coefficients $f_p(\xi)$ $(p = 0, 1, \ldots)$ are

$$f_0^{cyl}(\xi) = -\frac{1}{6}\ln\frac{2\,\theta_2^2}{\theta_4\theta_3}, \quad (88)$$

$$f_1^{cyl}(\xi) = -\frac{\pi^3\xi^2}{360}\left(\frac{7}{8}\theta_2^8-\theta_3^4\theta_4^4\right), \quad (89)$$

$$f_p^{cyl}(\xi) = \frac{\pi^{2p+1}\xi^{p+1}}{(2p+2)(2p)!}\Lambda_{2p}\,K_{2p+2}^{0,\frac{1}{2}}(i\xi) \quad \text{for} \quad p=2,3,\ldots \quad (90)$$

$$f_0^{Mob}(\xi) = -\frac{1}{6}\ln\frac{2\,\theta_3^2}{\theta_2\theta_4}, \quad (91)$$

$$f_1^{Mob}(\xi) = -\frac{\pi^3\xi^2}{360}\left(\frac{7}{8}\theta_3^8+\theta_2^4\theta_4^4\right), \quad (92)$$

$$f_p^{Mob}(\xi) = \frac{\pi^{2p+1}\xi^{p+1}}{(2p+2)(2p)!}\Lambda_{2p}\,K_{2p+2}^{\frac{1}{2},\frac{1}{2}}(i\xi) \quad \text{for} \quad p=2,3,\ldots \quad (93)$$

Now using the expressions for the partition functions $Z_{2M,2N-1}^{torus}, Z_{2M,2N-1}^{free}, Z_{2M,2N-1}^{cyl}, Z_{4M,2N-1}^{Klein}$ and $Z_{4M,2N-1}^{Mob}$ given by equations (58)–(62) one can easily obtain the expansion coefficients $f_p(\xi)$ $(p = 0, 1, \ldots)$ for periodic, free and Klein bottle boundary conditions through the following functional relations

$$f_p^{torus}(\xi) = 2^{p+1}\,f_p^{cyl}(\xi/2), \quad (94)$$

$$f_p^{free}(\xi) = f_p^{cyl}(\xi) + \frac{\delta_{p,0}}{4}\ln(8z_v^2), \quad (95)$$

$$f_p^{Klein}(\xi) = 2^{p+1}\,f_p^{Mob}(\xi/2). \quad (96)$$

Thus we derive exact finite-size corrections for the logarithm of the partition function of the dimer model on the $M\times N$ square lattice with five different sets of boundary conditions and have found that the exact asymptotic expansion of the free energy of the dimer model can be written in the form given by equation (1). Except the bulk free energy f_{bulk} all other coefficients in this expansion are sensitive to the boundary conditions. The surface contribution to the free energy f_{1s} and f_{2s} depend on the type of boundary conditions but independent on the parities (even or odd) of M and N. The other coefficients $f_p(\xi)$ $(p = 0, 1, \ldots)$ in the expansion of equation (1) show strong dependence as on parity of the lattice side M and N as well on the boundary conditions.

2.3 The spanning tree model on square lattice under different boundary conditions

Enumeration of spanning trees on a graph is a classical problem of combinatorial graph theory, first considered by Kirchhoff [100] in his analysis of electrical networks. Let $G = V, E$ denote a connected graph (without loops) with vertex and edge sets V and E. A spanning subgraph of G is a spanning tree T if it has $V1$ edges with at least one edge incident at each vertex. The degree of a vertex is the number of edges attached to it (often denoted coordination number). According to the Kirchhoff theorem, the number of spanning-tree subgraphs on a lattice is given by the minors of the discrete Laplacian matrix Δ of this lattice. The Laplacian matrix Δ is defined as

$$\Delta = Q - A, \tag{97}$$

where A is an $N_s \times N_s$ adjacency matrix, and N_s is the number of lattice sites. The elements of matrix A are given by $A_{ij} = 1$, if sites i and j are adjacent, $A_{ij} = 0$, otherwise, and Q is an $N_s \times N_s$ degree matrix of G with elements $Q_{ij} = k_i \delta_{ij}$, where k_i is the degree of site i, and δ_{ij} is the Kronecker δ function. In 2000 Tzeng and Wu [101] obtained the closed-form expressions for the spanning tree generating function for a hypercubic lattice in d dimensions under free, periodic, and a combination of free and periodic-boundary conditions. They also obtained the spanning-tree-generating function for a simple quartic net embedded on two nonorientable surfaces, a Mobius strip and the Klein bottle.

In 2015 Izmailian and Kenna [39] express the partition functions of the spanning tree on finite square lattices under five different boundary conditions (free, cylindrical, toroidal, Moebius strip, and Klein bottle) in terms of a principal partition function with twisted boundary conditions. Based on these expressions, they derive the exact asymptotic expansions of the logarithm of the partition function for all boundary conditions mentioned above. They have also established several groups of identities relating spanning tree partition functions for the different boundary conditions.

Let us consider the problem of enumerating weighted spanning trees on the $M \times N$ rectangular lattice. The problem of enumerating spanning trees on a graph was first considered by Kirchhoff in his analysis of electrical networks [100]. The enumeration of spanning trees involves the evaluation of the tree generating (or partition) function

$$Z_{M,N} = \sum_T x_1^{n_h} x_2^{n_v}, \tag{98}$$

where weights x_1 and x_2, respectively, are assigned to the edges in the horizontal and vertical direction. The summation is taken over all spanning tree configurations T on the lattice and, n_h and n_v are the numbers of edges in the spanning tree in the respective directions. Without lose the generality we will consider the case $x_1 = x_2$. It has been shown [39] that the partition function of the spanning tree on $M \times N$ lattice is expressed in terms of the principal partition function with twisted boundary conditions $Z_{0,0}(M, N, 0, 0)$ and $Z_{\frac{1}{2},\frac{1}{2}}(M, N, 0, 0)$ only, namely

$$Z^{torus}_{M,N} = \frac{1}{MN} Z^2_{0,0}(M, N, 0, 0), \tag{99}$$

$$Z^{cyl}_{M,N} = \frac{1}{2N \sinh{(Ma)}} Z_{0,0}(M, 2N, 0, 0), \tag{100}$$

$$Z^{free}_{M,N} = \frac{2^{1/4}}{\sqrt{2MN \sinh{(2Na)} \sinh{(2M\,a)}}} \times Z^{1/2}_{0,0}(2M, 2N, 0, 0), \tag{101}$$

$$Z^{Mob}_{M,N} = \frac{1}{2M|\sinh{\left(Na + \frac{i\pi M}{2}\right)}|} Z_{0,0}(M, N) \times Z_{\frac{1}{2},\frac{1}{2}}(M, N, 0, 0), \tag{102}$$

$$Z^{Klein}_{M,N} = \begin{cases} \frac{\coth{(N\,a)}}{2M} Z_{0,0}(M, 2N, 0, 0), & \text{for even } M \\ \frac{1}{2M} Z_{0,0}(M, 2N, 0, 0), & \text{for odd } M. \end{cases} \tag{103}$$

It is clear that $Z_{0,0}(M, N, 0, 0)$ given by equation (17) for the case $d = 0$ and $\mu = 0$ vanishes due to the zero mode at $(m, n) = (0, 0)$. In what follows, therefore, we remove the zero mode, and when $\alpha = \beta = 0$ replace $Z_{0,0}(M, N, 0, 0)$ in equation (17) by

$$Z^2_{0,0}(M, N, 0, 0) = \prod_{m=0}^{M-1} \prod_{n=0}^{N-1}{}' 4\left[\sin^2 \tfrac{m\pi}{M} + \sin^2 \tfrac{n\pi}{N}\right], \tag{104}$$

where the prime on the product denotes the restriction $(m, n) \neq (0, 0)$. The general theory for the asymptotic expansion of $Z_{\alpha,\beta}(M, N, 0, 0)$ for $(\alpha, \beta) \neq (0, 0)$ has been given in [48]. The exact asymptotic expansion of the $\ln Z_{0,0}(M, N, 0, 0)$ in terms of the Kronecker's double series $K^{0,0}_{2p+2}(i\xi)$ has been obtained in (see Appendix A of [78])

$$\ln Z_{0,0}(M, N, 0, 0) = \frac{S}{\pi} \int_0^\pi \omega_0(x) dx + \ln \sqrt{S\xi} + 2 \ln \eta(i\xi) - 2\pi\xi \sum_{p=1}^{\infty} \left(\frac{\pi^2 \xi}{S}\right)^p \frac{\Lambda_{2p}}{(2p)!} \frac{K^{0,0}_{2p+2}(i\xi)}{2p+2}, \tag{105}$$

where S is the area of the lattice, ξ is the aspect ratio, $\eta(i\xi)$ is Dedeking η-function, $\omega_0(x)$ is given by equation (19) for the case $\mu = 0$ and Λ_{2p} is given by equation (29).

2.3.1 Exact finite-size corrections of the free energy for spanning tree model on square lattice under different boundary conditions

The exact asymptotic expansions of the free energy for all boundary conditions can be written in the form given by equation (1). The bulk free energy $f_{\rm bulk}$ for the spanning tree on an $M \times N$ lattices is the same for all boundary conditions and given by

$$f_{bulk} = -\frac{4G}{\pi}, \tag{106}$$

where G is the Catalan constant.

Spanning tree on the torus

For the spanning tree on the torus the surface free energy for the spanning tree f_{1s} and f_{2s} in equation (1) are equal to zero. For the leading correction terms $f_0(\xi)$ we obtain

$$f_0(z\xi) = -\ln\xi - 4\ln\eta(iz\xi) = -2\ln\eta(iz\xi)\eta(i/(z\xi)), \tag{107}$$

for subleading correction terms $f_p(\xi)$ for $p = 1, 2, 3, \ldots$, we obtain

$$f_p(\xi) = 4\pi^{2p+1}\xi^{p+1}\frac{\Lambda_{2p}}{(2p)!}\frac{K^{0,0}_{2p+2}(i\xi)}{2p+2}. \tag{108}$$

The coefficients Λ_{2p} are listed in equation (29) and Kronecker's double series $K^{0,0}_{2p+2}$ in terms of the elliptic theta functions are given in Appendix B of [78] for $p = 1, 2, 3$ and 4.

It is easy to see from equation (107), that for the spanning tree on finite square lattices under periodic boundary conditions, $f_0(\xi)$ does not contain the corner free energy f_{corner} given by equation (4), which confirm both conformal theory [66] and finite-size scaling [102] predictions that logarithmic corner corrections to the free energy density should be absent for periodic boundary conditions. However, such terms have been found by Duplantier and David [79] in the two-dimensional spanning tree (ST) model under periodic boundary conditions

$$f_{\text{corner}} = -\ln S. \tag{109}$$

This discrepancy comes from the fact that equation (109) has been obtained for the rooted spanning tree model. The logarithmic correction to the free energy obtained by Duplantier and David equation (109) is connected with the fact that number of rooted spanning trees is S times larger than that of the un-rooted spanning trees (see Eq. (1.3) of [79]). It is not related to the contribution to free energy from the corner. Taking into account that the result for the free energy for un-rooted spanning trees differs from the rooted spanning trees by a factor $\ln S$, we can obtain the correct version for the corner free energy $f_{\text{corner}} = 0$ by adding to equation (109) the term $\ln S$.

Spanning tree on the plane

For the spanning tree on the plane the surface free energy f_{1s} and f_{2s} are given by

$$f_{1s} = \frac{1}{2}\ln(1+\sqrt{2}), \tag{110}$$

$$f_{2s} = \frac{1}{2}\ln(1+\sqrt{2}). \tag{111}$$

For the leading correction terms $f_0(z\xi)$ we obtain

$$f_0(\xi) = \frac{1}{4}\ln S - \frac{1}{2}\ln\eta(i\xi)\eta(i/\xi) - \frac{5}{4}\ln 2. \tag{112}$$

For subleading correction terms $f_p(\xi)$ for $p = 1, 2, 3, \ldots$, we obtain

$$f_p(\xi) = \frac{\pi^{2p+1}\xi^{p+1}}{2^{2p}}\frac{\Lambda_{2p}}{(2p)!}\frac{K^{0,0}_{2p+2}(i\xi)}{2p+2}.$$

It is easy to see from equation (112) that for the spanning tree on finite square lattices under free boundary condition $f_0(\xi)$ contains the universal part f_{univ} given by equation (8). This confirms the conformal field theory prediction for the corner free energy in models for which the central charge is $c = -2$,

$$f_{\text{corner}} = \frac{1}{4}\ln S. \tag{113}$$

Moreover, $f_0(\xi)$ contains the non-universal, geometry-independent constant $f_{\text{nonuniv}} = -\frac{5}{4}\ln 2$. Again, as in the case of periodic boundary conditions, there is discrepancy with the results of Duplantier and David [79] for the corner free energy in the spanning tree on finite square lattices under free boundary condition. They obtained for the corner free energy the expression

$$f_{\text{corner}} = -\frac{3}{4}\ln S \tag{114}$$

which is different from the conformal field theory prediction equations (4) and (113). Noting that the result for the un-rooted spanning tree differs from that of the rooted spanning tree by a factor $\ln S$, we obtain the correct version for the corner free energy given by equation (4) with $c = -2$ by adding to equation (114) the term $\ln S$.

Spanning tree on the cylinder

For the spanning tree on the cylinder the surface free energy f_{1s} is given by equation (110) and f_{2s} is equal to zero. For the leading correction terms $f_0(\xi)$ we obtain

$$f_0(\xi) = -\ln\eta(i\xi)\eta(i/(\xi)) - \frac{1}{2}\ln 2 - \frac{1}{2}\ln\xi, \tag{115}$$

for subleading correction terms $f_p(\xi)$ for $p = 1, 2, 3, \ldots$, we obtain

$$f_p(\xi) = \frac{\pi^{2p+1}\xi^{p+1}}{2^{p-1}}\frac{\Lambda_{2p}}{(2p)!}\frac{K^{0,0}_{2p+2}(i\xi)}{2p+2}.$$

Spanning tree on the Möbius strip

For the spanning tree on the Möbius strip the surface free energy f_{1s} is given by equation (110) and f_{2s} in equal to zero. For the leading correction terms $f_0(\xi)$ we obtain

$$f_0(\xi) = -\ln\xi - \ln\theta_3(i\xi)\eta(i\xi), \tag{116}$$

for subleading correction terms $f_p(\xi)$ for $p = 1, 2, 3, \ldots$, we obtain

$$f_p(\xi) = 2\pi^{2p+1}\xi^{p+1}\frac{\Lambda_{2p}}{(2p)!}\frac{K^{0,0}_{2p+2}(i\xi) + K^{\frac{1}{2},\frac{1}{2}}_{2p+2}(i\xi)}{2p+2}.$$

Kronecker's double series $K^{\frac{1}{2},\frac{1}{2}}_{2p+2}(i\xi)$ in terms of the elliptic theta functions are given in [23] for $p = 1, 2$ and in [48] for $p = 3, 4$.

Spanning tree on the Klein bottle

For the spanning tree on the Klein bottle the surface free energy f_{1s} and f_{2s} are equal to zero. For the leading correction terms $f_0(\xi)$ we obtain

$$f_0(\xi) = -\ln 2\xi - 2\ln \eta(2i\xi), \tag{117}$$

for subleading correction terms $f_p(\xi)$ for $p = 1, 2, 3, \ldots$, we obtain

$$f_p(\xi) = 4\pi^{2p+1}\xi^{p+1}\frac{\Lambda_{2p}}{(2p)!}\frac{K^{0,0}_{2p+2}(2i\xi)}{2p+2}.$$

In this subsection we derive exact finite-size corrections for the logarithm of the partition function of the spanning-tree model on the $M \times N$ square lattice with five different sets of boundary conditions. We have found that the exact asymptotic expansion of the free energy of the spanning-tree model can be written in the form given by equation (1). Except the bulk free energy f_{bulk} all other coefficients in this expansion are sensitive to the boundary conditions. We explain an apparent discrepancy between conformal field theory predictions and a two dimensional spanning tree model with periodic and free boundary conditions [79,80]. We have also obtained the corner free energy for free boundary conditions. We proved the conformal field theory prediction about the corner free energy and have shown that the corner free energy, which is proportional to the central charge c, is indeed universal. We find the central charge in the framework of the conformal field theory to be $c = -2$.

2.4 Resistor network on a M × N lattice

The calculation of the resistance between arbitrary node of infinite networks of resistors is a well studied subject [103–105]. Resistor networks have been widely studied as models for conductivity problems and classical transport in disordered media [106–108]. Besides being a central problem in electric circuit theory, the computation of resistances is also relevant to a wide range of problems ranging from random walks (see [104] and [109], and discussions below), first-passage processes [110], to lattice Green's functions [111]. Past studies have been focused mainly on infinite lattices [112,113]. Little attention has been paid to finite network, even though the latter are those occurring in real life. Recently, Wu [17] has revisited the two-point resistance problem and deduced a closed-form expression for the resistance between arbitrary two nodes for finite networks with resistors r and s in the two spatial directions. Later, Jafarizadeh et al. [114] proposed an algorithm for the calculation of the resistance between two arbitrary nodes in an arbitrary distance-regular networks. However, the exact expression obtained in [17] is in the form of a double summation whose mathematical and physical contents are not immediately apparent. Quite recently Essam and Wu based on the exact expression for the resistance between arbitrary two nodes for finite rectangular network obtained in [49] has derived the asymptotic expansion for the corner-to-corner resistance ($R^{\rm free}_{M,N}(r,s)$) on an $M \times N$ rectangular resistor network under free boundary conditions. For the case $M = N$ and $r = s = 1$ they computed the finite-size corrections to the corner-to-corner resistance up to order N^{-4}:

$$R^{\rm free}_{N,N}(1,1) = \frac{4}{\pi}\ln N + 0.077318 + \frac{0.266070}{N^2} - \frac{0.534779}{N^4} + O\left(\frac{1}{N^6}\right).$$

The computation of the asymptotic expansion of the corner-to-corner resistance (in other word the resistance between two maximally separated nodes) of a rectangular resistor network has been of interest for some time, as its value provides a lower bound to the resistance of compact percolation clusters in the Domany-Kinzel model of a directed percolation [115].

In 2010 Izmailian and Huang [36] derive the exact asymptotic expansions for resistance between two maximally separated nodes on the rectangular network under free, periodic and cylindrical boundary conditions. They show that the exact asymptotic expansion of the resistance between nodes of the network for all boundary conditions can be written in the form given by equation (9). The all coefficients in this expansion are expressed through analytical functions. Based on the exact expression for the resistance between arbitrary two nodes for finite rectangular network under free, periodical and cylindrical boundary conditions obtained in [17] they express the resistance between two most separated nodes in terms of $G_{\alpha,\beta}(\rho, M, N)$ with $(\alpha, \beta) = (1/2, 0)$ and $(0, 1/2)$. They then extend Ivashkevich et al. [23] to derive the exact asymptotic expansions of the resistance between two maximally separated nodes on the rectangular network for all boundary conditions and write down the expansion coefficients up to the second order. Recently, using the same approach Izmailian and Kenna [72] derive the asymptotic expansions for the resistance between the center node and a node on the boundary of the $M \times N$ cobweb network with resistors r and s in the two spatial directions.

An electrical network can be regarded as a graph in which the resistance R_{ij} is associated to the edge between pair of connected nodes i and j. Denote the electric potential at the ith vertex by V_i and the net current flowing into the network at the ith vertex by I_i. When the potential difference occurs between points i and j, the current is given by the Ohm's law $I_{ij} = (V_i - V_j)C_{ij}$, where $C_{ij} = 1/R_{ij}$ is the conductance of the respective link. By the Kirchhoff's current law total current outflow from any point in the interior is zero, $\sum_j I_{ij} = 0$, we then find for the voltage

$$V_i = \sum_j V_j C_{ij}/C_i \tag{118}$$

where $C_i = \sum_j C_{ij}$ and the sum is over all nodes j which are connected to i.

The two-point resistance has a probabilistic interpretation based on classical random walker walking on the network. The averaging property expressed by equation (118) implies that the voltage is a harmonic function on the interior points of the graph. This makes the basis for the probabilistic interpretation of the voltage [104,116–118]. The random walk determined by the electrical network is defined as finite state Markov chain (for more details see [104]) with the transition probabilities P_{ij} that are weighted with the conductances as $P_{ij} = C_{ij}/C_i$. Then, when the constant voltage is applied to the graph such that $V_a = 1$ and $V_b = 0$, the voltage in an interior point x is determined as the hitting probability h_x that a walker staring at x reaches the point a before reaching b.

2.4.1 The exact asymptotic expansion for the resistance between two maximally separated nodes of the resistor network under free, periodic and cylindrical boundary conditions

Consider a rectangular $M \times N$ network of resistors with resistances r and s on edges of the network in the respective horizontal and vertical directions. The closed-form expression for the resistance R_{ij} between arbitrary two nodes $i = (x_1, y_1)$ and $j = (x_2, y_2)$ for free, periodic and cylindrical boundary conditions was obtained in [17].

In [71] it has been shown that the resistance $R_{M,N}(r, s)$ between two maximally separated nodes of the network for all above mentioned boundary conditions can be expressed in terms of $G_{\alpha,\beta}(\rho, M, N)$ only,

$$R^{\rm free}_{M,N}(r,s) = -r + \frac{\sqrt{rs}}{S}\left(G_{0,1/2}(\rho, M, N) + G_{1/2,0}(\rho, M, N)\right), \tag{119}$$

$$R^{\rm per}_{M,N}(r,s) = \frac{\sqrt{rs}}{S}\left(G_{0,1/2}(\rho, M/2, N/2) + G_{1/2,0}(\rho, M/2, N/2)\right), \tag{120}$$

$$R^{\rm cyl}_{M,N}(r,s) = \frac{\sqrt{rs}}{S}\left(G_{0,1/2}(\rho, M/2, N) + G_{1/2,0}(\rho, M/2, N)\right), \tag{121}$$

where ρ is given by equation (10) and $G_{\alpha,\beta}(\rho, M, N)$ is given by

$$G_{\alpha,\beta}(\rho, M, N) = M \;\; \mathtt{Re} \sum_{n=0}^{N-1} f\left(\pi\frac{n+\alpha}{N}\right) \times \coth\left[M\,\omega\left(\pi\frac{n+\alpha}{N}\right) + i\pi\beta\right] \tag{122}$$

for $(\alpha, \beta) \neq (0, 0)$. The function $\omega(x)$ is the same for all boundary conditions and given by:

$$\omega(x) = \sqrt{\rho}\sin x \tag{123}$$

and function $f(x)$ is depend on boundary conditions and given by

$$f(x) = \frac{\cos^2 x\sqrt{1+\rho\sin^2 x}}{\sin x} \qquad \text{for free BCs,} \tag{124}$$

$$f(x) = \frac{1}{\sin x\,\sqrt{1+\rho\sin^2 x}} \qquad \text{for periodic BCs,} \tag{125}$$

$$f(x) = \frac{\cos^2 x}{\sin x\,\sqrt{1+\rho\sin^2 x}} \qquad \text{for cylindrical BCs.} \tag{126}$$

It is easy to see that $G_{\alpha,\beta}(\rho, M, N)$ can be consider as particular case of the second derivative of the logarithm of the partition function with twisted boundary conditions $Z_{\alpha,\beta}(M, N, \mu, d)$ given by equation (18) at $\mu = 0, d = 0$. To see that let us take the second derivative of equation (18) at $d = 0$ with respect to mass variable μ and then consider limit $\mu \to 0$. As result we obtain

$$\frac{Z''_{\alpha,\beta}(M,N,0,0)}{Z_{\alpha,\beta}(M,N,0,d)} = M \;\; \mathtt{Re}\sum_{n=0}^{N-1}\omega''_0\left(\tfrac{\pi(n+\alpha)}{N}\right) \times \coth\left[M\omega_0\left(\tfrac{\pi(n+\alpha)}{N}\right) + i\pi\beta\right] \tag{127}$$

where $\omega''_0(x)$ is the second derivative of $\omega_\mu(x)$ with respect to μ at criticality. Now using in equation (127) instead of $\omega''_0(x)$ the function $f(x)$ given by equations (124)–(126) and instead of $\omega_0(x)$ the function $\omega(x)$ given by equation (123) we can see that the second derivative of the logarithm of the partition function with twisted boundary conditions $Z''_{\alpha,\beta}(M, N, 0, 0)$ at $\mu = 0, d = 0$ given by equation (127) is exactly coincide with expression for $G_{\alpha,\beta}(\rho, M, N)$ given by equation (122). Thus the exact asymptotic expansion of the $G_{\alpha,\beta}(\rho, M, N)$ for $(\alpha, \beta) = (0, 1/2), (1/2, 0)$ can be found along the same lines as in the case of $Z''_{\alpha,\beta}(M, N, 0, 0)$ (see [23])

$$G_{\alpha,\beta}(\rho,M,N) = \frac{2\kappa S}{\pi}\left[\ln\sqrt{\frac{S}{\xi}} + C_E + \ln\frac{2^{5/2}}{\pi} - 2\ln|\theta_{\alpha,\beta}|\right] - \kappa\pi\xi\sum_{p=1}^{\infty}\frac{1}{p(2p)!}\left(\frac{\pi^2\xi}{S}\right)^{p-1} \mathtt{Re}\;\Omega_{2p}\,K^{\alpha,\beta}_{2p}, \tag{128}$$

where C_E is the Euler constant.

The differential operators Ω_{2p} that have appeared here can be expressed via coefficients $\omega_{2p} = \varepsilon_{2p} + \lambda_{2p}\frac{\partial}{\partial\tau_0}$ as

$$\Omega_2 = \omega_2, \quad \Omega_4 = \omega_4 + 3\omega_2^2, \quad \vdots \tag{129}$$

and the coefficients k and ε_{2p} can be obtain from the asymptotic expansion of the $f(x)$, which can be written in the following form

$$f(x) = \frac{\kappa}{x}\left[1 + \sum_{p=1}^{\infty}\frac{\kappa_{2p}}{(2p)!}x^{2p}\right] = \frac{\kappa}{x}\;\exp\left\{\sum_{p=1}^{\infty}\frac{\varepsilon_{2p}}{(2p)!}x^{2p}\right\} \tag{130}$$

where the coefficients ε_{2p} and κ_{2p} are related to each other through relation between moments and cumulants [23].

After reaching this point, one can easily write down all the terms of the exact asymptotic expansion for the resistance between two maximally separated nodes $R_{M,N}(r,s)$ on an $M\times N$ rectangular network of resistors with various boundary conditions using equation (128). Thus the exact asymptotic expansion for the resistance between two maximally separated nodes can be written in the form given by equation (9). The coefficients $c(\rho)$ and $c_{2p}(\rho,\xi)$ $(p=0,1,2,\ldots)$ in the expansion equation (9).

1. Free boundary conditions.

For a rectangular $M\times N$ network of resistors with free boundary conditions the coefficients $c(\rho)$ and $c_{2p}(\rho,\xi)$ $(p=0,1,2,\ldots)$ in the expansion equation (9) explicitly given by

$$c(\rho) = \frac{2\sqrt{\rho}}{\pi} \quad (131)$$

$$c_0(\rho,\xi) = \frac{2\sqrt{\rho}}{\pi}\left(2\ln\frac{8}{\pi}+2C_E-1-\ln\xi(1+\rho)-\frac{\pi\sqrt{\rho}}{2} + \frac{\rho-1}{\sqrt{\rho}}\arctan\sqrt{\rho}-2\ln\theta_2(i\sqrt{\rho}\,\xi)\theta_4(i\sqrt{\rho}\,\xi)\right) \quad (132)$$

$$c_{2p}(\rho,\xi) = \frac{\pi^{2p-1}\xi^p\sqrt{\rho}}{p(2p)!}\Omega_{2p}\Big[K_{2p}^{0,1/2}(i\sqrt{\rho}\,\xi)+K_{2p}^{1/2,0}(i\sqrt{\rho}\,\xi)\Big] \quad \text{for} \quad p=1,2,\ldots \quad (133)$$

where the differential operators Ω_{2p} is given by equation (129) and $K_{2p}^{0,1/2}(i\sqrt{\rho}\xi)$, $K_{2p}^{1/2,0}(i\sqrt{\rho}\xi)$ are the Kronecker's double series which can all be expressed in terms of the elliptic $\theta_k(i\sqrt{\rho}\xi)$ $(k=2,3,4)$ functions only (see [23,48]).

2. Periodic boundary conditions.

For a rectangular $M\times N$ network of resistors with periodic boundary conditions the coefficients $c(\rho)$ and $c_{2p}(\rho,\xi)$ $(p=0,1,2,\ldots)$ in the expansion equation (9) explicitly given by

$$c(\rho) = \frac{\sqrt{\rho}}{2\pi} \quad (134)$$

$$c_0(\rho,\xi) = \frac{\sqrt{\rho}}{2\pi}\left(2\ln\frac{4}{\pi}+2C_E-\ln\xi(1+\rho)-2\ln\theta_2\theta_4\right) \quad (135)$$

$$c_{2p}(\rho,\xi) = \frac{4^{p-1}\pi^{2p-1}\xi^p\sqrt{\rho}}{p(2p)!}\Omega_{2p} \times\left[K_{2p}^{0,1/2}(i\sqrt{\rho}\,\xi)+K_{2p}^{1/2,0}(i\sqrt{\rho}\,\xi)\right] \quad \text{for} \quad p=1,2,\ldots \quad (136)$$

3. Cylindrical boundary conditions.

For a rectangular $M\times N$ network of resistors with cylindrical boundary conditions the coefficients $c(\rho)$ and $c_{2p}(\rho,\xi)$ $(p=0,1,2,\ldots)$ in the expansion equation (9) explicitly given by

$$c(\rho) = \frac{\sqrt{\rho}}{\pi} \quad (137)$$

$$c_0(\rho,\xi) = \frac{\sqrt{\rho}}{\pi}\left(2\ln\frac{8}{\pi}+2C_E-\ln\xi(1+\rho) -\frac{2}{\sqrt{\rho}}\arctan\sqrt{\rho}-2\ln\theta_2\theta_4\right) \quad (138)$$

$$c_{2p}(\rho,\xi) = \frac{\pi^{2p-1}\xi^p\sqrt{\rho}}{2p(2p)!}\Omega_{2p}\Big[K_{2p}^{0,1/2}(i\sqrt{\rho}\,\xi)+K_{2p}^{1/2,0}(i\sqrt{\rho}\,\xi)\Big] \quad \text{for } p=1,2,\ldots \quad (139)$$

2.4.2 The exact asymptotic expansion for the resistance between the central node and a node on the boundary of the cobweb network

Let us consider the cobweb network. The cobweb lattice $\mathcal{L}_{\rm cob}$ is an $M\times N$ rectangular lattice with periodic boundary condition in one direction and nodes on one of the two boundaries in the other direction connected to an external common node. Therefore there is a total of $MN+1$ nodes. Topologically $\mathcal{L}_{\rm cob}$ is of the form of a wheel consisting of N spokes and M concentrate circles. There has been considerable recent interest in studying the resistance in a cobweb and in a cobweb-like networks (see for example references [119–124]).

The closed-form expression for the resistance between arbitrary two nodes (x_1,y_1) and (x_2,y_2) for cobweb network was obtained in [119]. In [72] it has been shown that the resistance $R^{\rm cob}_{M,N}(r,s)$ between the central node $O=(0,0)$ and node on the boundary of the network $A=(x,M)$ can be expressed in terms of $G_{0,1/2}(\rho,M,N)$ only, namely

$$R^{\rm cob}_{M,N}(r,s) = -\frac{s}{2}+\frac{\sqrt{sr}}{4S}G_{0,1/2}(\rho,2M+1,N) \quad (140)$$

where $S=(M+1/2)N$ and $G_{0,1/2}(\rho,2M+1,N)$ is given by equation (122) with $\alpha=0,\beta=1/2$ and M replaced by $2M+1$

$$G_{0,1/2}(\rho,2M+1,N) = (2M+1)\ \text{Re}\sum_{n=0}^{N-1}f\left(\frac{\pi n}{N}\right) \times\coth\left[(2M+1)\,\omega\left(\frac{\pi n}{N}\right)+\frac{i\pi}{2}\right] \quad (141)$$

$$= (2M+1)\ \text{Re}\sum_{n=0}^{N-1}f\left(\frac{\pi n}{N}\right)\ \tanh\left[(2M+1)\,\omega\left(\frac{\pi n}{N}\right)\right]. \quad (142)$$

The function $\omega(x)$ is given by equation (123) and function $f(x)$ is given by

$$f(x) = \frac{\sqrt{1+\rho\sin^2 x}}{\sin x}. \quad (143)$$

After reaching this point, using equation (128) one can easily write down all the terms of the exact asymptotic expansion for the resistance between the central node and a node on the boundary of the $M \times N$ cobweb network of resistors $R^{\rm cob}_{M,N}(r,s)$. Thus the exact asymptotic expansion for the resistance $R^{\rm cob}_{M,N}(r,s)$ can be written in the form given by equation (9). The coefficients $c(\rho)$ and $c_{2p}(\rho,\xi)$ $(p = 0,1,2,\ldots)$ in the expansion equation (9) explicitly given by

$$c(\rho) = \frac{\sqrt{\rho}}{2\pi} \tag{144}$$

$$c_0(\rho,\xi) = \frac{\sqrt{\rho}}{2\pi}\left(2\ln\frac{8}{\pi} + 2C_E - 1 - \ln\xi(1+\rho) + \frac{2}{\sqrt{\rho}}\arctan\sqrt{\rho} - 4\ln\theta_2(i\sqrt{\rho}\,\xi)\right) \tag{145}$$

$$c_{2p}(\rho,\xi) = -\frac{\pi^{2p-1}\xi^p\sqrt{\rho}}{2p(2p)!}\varOmega_{2p}\,K^{0,1/2}_{2p}(i\sqrt{\rho}\,\xi) \quad \text{for} \quad p = 1,2,3,\ldots \tag{146}$$

where $\xi = (2M+1)/N$, which is almost two times larger then true aspect ratio of the lattice $\xi_{true} = M/N$. The differential operators $\varOmega_{2p}$ is given by equation (129) and $K^{0,1/2}_{2p}(i\sqrt{\rho}\,\xi)$ is the Kronecker's double series which can all be expressed in terms of the elliptic $\theta_k(i\sqrt{\rho}\,\xi)$ $(k = 2,3,4)$ functions only.

3 Conclusions

We have consider the exact finite-size corrections and boundary effects for the critical two-dimensional free-fermion lattice models. We have shown that the method of references [23,45] is quite useful for calculating exact finite-size corrections for the free models of statistical mechanics, including Ising model, dimer model, resistor network and spanning tree model. In this review article we tries to unify the different results on exact finite-size corrections to a common framework. The main results, which is reviewed in this review article can be summarized as follow:

We have found that the exact asymptotic expansion of the free energy can be written in the form given by equation (1) for the following models:

(i) For the Ising model on the $M \times N$ square lattice with three different boundary conditions: periodic, helical and Brascamp-Kunz.

(ii) For the dimer model on the $M \times N$ square lattice with five different sets of boundary conditions: periodic, cylindrical, free Mobius strip and Klein bottle.

(iii) For the spanning-tree model on the $M \times N$ square lattice with five different sets of boundary conditions. We have explained an apparent discrepancy between conformal field theory predictions and a two dimensional spanning tree model with periodic and free boundary conditions [79,80]. We have also obtained the corner free energy for free boundary conditions. We proved the conformal field theory prediction about the corner free energy and have shown that the corner free energy, which is proportional to the central charge c, is indeed universal. We find the central charge in the framework of the conformal field theory to be $c = -2$.

Except the bulk free energy f_{bulk} all other coefficients in the exact asymptotic expansion of the free energy of the Ising model, dimer model and spanning tree model are sensitive to the boundary conditions.

We have also study the two-point resistor problem on planar $M \times N$ rectangular lattices with free, periodic and cylindrical boundary conditions. Using the exact expression for the resistance between arbitrary two nodes for finite rectangular network obtained in [17] and the algorithm of [23], we derive the exact asymptotic expansion of the corner-to-corner resistance on the rectangular network for all above mentioned boundary conditions. We have found that the exact asymptotic expansion can be written in the form given by equation (9). We have found that all coefficients in this expansion are sensitive to the boundary conditions. All corrections to scaling are analytic.

It is of interest to apply the method of references [23,45] to calculate exact finite-size corrections for other exactly solvable models on various lattices with various boundary conditions, so that some general features of such finite-size corrections could be found.

The author thanks Applied Mathematics Research Centre at the Coventry University, for hospitality during completion of this work. This work was partially supported by IRSES (Projects No. 612707-DIONICOS) within 7th European Community Framework Programme and by a grant from the Science Committee of the Ministry of Science and Education of the Republic of Armenia under Contract No. 15T-1C068.

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